The localisation transition for directed polymers based on minicourse by H. Lacoin

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These notes are produced entirely from the minicourse I followed at the Probability at Warwick conference in July 2025, `https://warwick.ac.uk/fac/sci/statistics/news/patw_summer_school/`, and my subsequent thoughts. They are not necessarily an accurate representation of what was presented, and may have in places been substantially edited. All errors are almost surely mine. Please send any corrections to pkt28@cam.ac.uk.

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1 Introduction

In these notes we will consider a probabilistic model for studying the interaction between a simple random walk and ambient noise, namely, that of the directed polymer. It first appeared in the statistical mechanics literature as a model of one dimensional interfaces of disordered Ising magnets almost fourty years ago. Such models are also linked to universality of stochastic growth phenomena, nowadays known as the Kardar-Parisi-Zhang universality class.

In a completely homogeneous environent, the polymer is nothing but a simple random walk. In this case one has many classical theorems from probability, such as the central limit theorem that give us a lot of information about the macroscopic structure of this model. However, by contidering an in homogeneous random environment and tuning the intensities of each inhomogeneity by an 'inverse temperature' parameter, the polymers start to exhibit new statistical phenomena. The precise transition between the classical 'diffusive' regime and other non-gaussian regimes has been subject to intense study over past decades. See [Zyg24] for a comprehensive review of directed polymers in random environments.

In these notes, we will focus on the results of [JL24] on the localisation transition for directed polymers in random environments in dimensions $d \geq 3$. More precisely, we will sketch a proof of the existence of a phase transition given by a critical inverse temperature $\beta_c \in (0,\infty)$, such that for $\beta < \beta_c$, the polymer measure is diffusive, while for $\beta > \beta_c$, the polymer measure is more localised.

The reader is encouraged to take up the study of this rich subject further by consulting the references given throughout these notes.

2 The simple random walk

We now begin our discussion with perhaps the simplest example of a random polymer, namely the simple random walk (SRW) on the integer lattice $\mathbb{Z}^d$, $d \geq 1$. On bounded intervals, say $\{1, \cdots, N\}$ for some $N \geq 1$ (which we will also denote as $[\![1, N]\!]$), since the walk can only make steps of unit size, the space of all possible attainable paths

$$\Omega^N := \{(X_n)_{n=0}^N \in (\mathbb{Z}^d)^{N+1} : X_0 = 0 \text{ and for all } 1 \leq n \leq N \,, \|X_{n+1} - X_n\|_{\ell_1} = 1\} \,, \tag{2.1}$$

is finite. In particular, it has cardinality $|\Omega^N| = (2d)^N$, and the measure induced by the simple random walk on $\mathbb{Z}^d$, restricted to $0 \leq n \leq N$ is simple the uniform measure on Ω^N.

Now, how does one conceive of the SRW as a 'polymer'? Formally, for $N \geq 1$, one also keeps track of the time by considering the process $(n, X_n)_{n=0}^N \in (\mathbb{Z}_+ \times \mathbb{Z}^d)^{N+1}$, which is also known as the directed SRW.

There are two important facts about the SRW for us, namely,

① The SRW endpoint distribution is delocalised, that is for $N \geq 1$, $\mu_N(\cdot) = P_N(X_N \in \cdot)$,

$$\max_{x \in \mathbb{Z}^d} \mu_N(x) \asymp N^{-d/2} \,,$$

which implies in particular that

$$\lim_{N \to \infty} \|\mu_N\|_\infty = 0 \,.$$

In other words, asymptotically, there are is no concentration of measure around atoms.

② The SRW is diffusive. More precisely, for $N \gg 1$, $|X_N| \asymp N^{1/2}$.

Complimentary to the above properties of the SRW, there is a classical result, namely the convergence on paths of the simple random walk under Brownian scaling to a Brownian motion, also known as Donsker's theorem (see Figure1).

Theorem 2.1 (Donsker's invariance principle)**.** *Let $X_1, X_2, \cdots$ be iid $\mathbb{R}-$valued integrable random variables with law μ, such that $\mathbb{E}[X_1] = 0$, and variance $\sigma^2 \in (0,\infty)$. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $S_t = (1 - \{t\})S_{[t]} + \{t\}S_{[t]+1}$, where $\{t\} = t - [t]$ and $[t]$ is the integer part of $t \geq 0$. Now, define*

$$S_t^{(N)} = \frac{S_{tN}}{\sqrt{\sigma^2 N}}$$

for $0 \leq t \leq 1$. Then, $\left(S_t^{(N)}, 0 \leq t \leq 1\right)$ converges weakly to $(B_t, 0 \leq 1 \leq 1)$, that is to a standard Brownian motion. More explicitly, we have for all continuous (in the local uniform topology) and bounded functionals $F : C([0,1], \mathbb{R}) \to \mathbb{R}$

$$\mathbb{E}\left[F(S^{(N)})\right] \overset{N\to\infty}{\longrightarrow} \mathbb{E}[F(B)] \,.$$

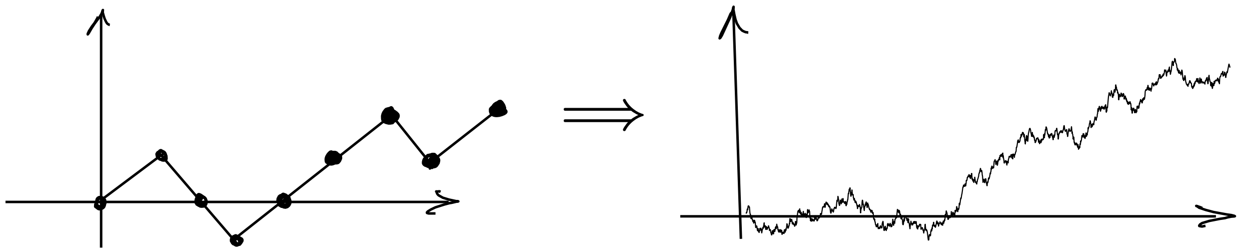

Figure 1: **Left**: a simple random walk on a finite interval. **Right**: a Brownian motion on another bounded interval. This figure is meant to illustrate Donsker's theorem: the convergence of the (linearly interpolated and rescaled) simple random walks to a Brownian motion. The convergence is established as one 'zooms out' in the left picture and appropriately rescales, and the resulting in the 'packing more and more' randomness on a given interval which establishes convergence with the aid of the strong law of large numbers.

3 Directed polymers in a random environments

For $N \geq 1$, one can think of the uniform (SRW) measures on the finite sample paths spaces (2.1) as those corresponding to a directed polymer in a completely homogeneous, or uniform environment. However, this can be generalised to environments with heterogeneities by adding weights to each location in the ambient lattice (see Figure 2) and consider exponential tilts of the uniform measure.

More precisely, we first sample independent and identically distributed weights, $\omega := (\omega_{n,x})_{n\geq 1, x\in\mathbb{Z}^d}$ known also as the 'random environment' or 'disorder'. For $N \geq 1$ and $X \in \Omega_N$ as in (2.1), define the 'energy' of a path $H_N^\omega(X) = \sum_{n=1}^N \omega_{n,X_n}$. Then, for $\beta \in [0, \infty)$, construct the (random) atomic measures

$$P_N^{\beta,\omega}(X) = \frac{\mathrm{e}^{-\beta H_N^\omega(X)}}{Z_N^{\beta,\omega}},$$

where the partition functions $Z_N^{\beta,\omega}$ (which correspond to normalisation constants for the tilted measures $P_N^{\beta,\omega}$) are given by

$$Z_N^{\beta,\omega} := \sum_{X\in\Omega_N} \mathrm{e}^{-\beta H_N^\omega(X)}. \tag{3.1}$$

The parameter β is also known in the literature as the "inverse temperature".

There are two extremal cases one can consider, corresponding to 'infinite temperature', of $\beta = 0$ or 'zero temperature', corresponding to $\beta = \infty$. In particular, one has

- (A) $\beta = 0$, $P_N^{\beta,\omega} = P_N \overset{n\to\infty}{\longrightarrow}$ SRW, by Theorem 2.1;
- (B) $\beta = \infty$, $P_N^{\beta,\omega} \overset{\beta\to\infty}{\longrightarrow} P_N^{\infty,\omega}$, where $P_N^{\infty,\omega}$ is almost surely supported on a single path (a Dirac mass).

The above gives some explanation as to why β is called an inverse temperature, with values of $\beta \in (0, \infty)$ interpolating between the 'frozen', localised polymers at $\beta = \infty$, and the 'thawed', diffusive polymers (directed SRW) in a 'heat bath infinite temperature' at $\beta = 0$.

4 Disorder in polymer environments

One hopes/expects there is a critical temperature $\beta_c \in (0, \infty)$, between the 'delocalisation' (like (A)) and 'localisation' (like (B)) regimes for the directed polymer measures (see Figure 3). A first, and perhaps not

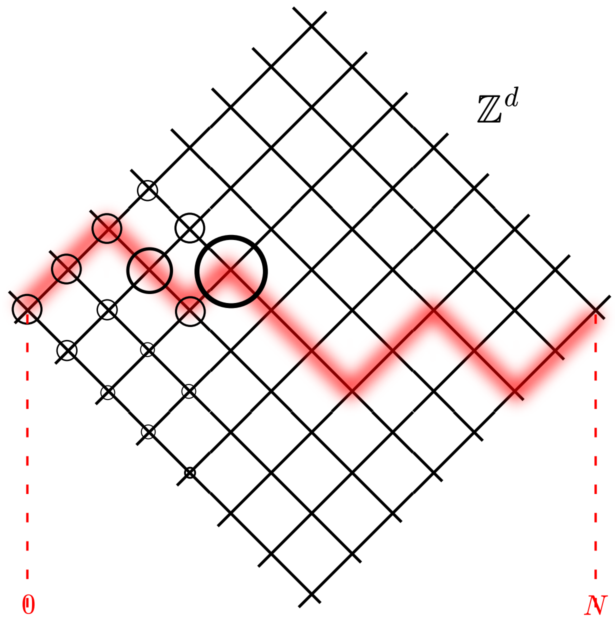

Figure 2: Illustration of a directed polymer (red) in a 'random environment' on the lattice $\mathbb{Z}^d$. where each circle denotes the strength of the weight at that location. The polymer measure favours paths with larger weights.

too surprising observation is that for finite values of β, one cannot have complete localisation of the polymer measures $P_N^{\beta,\omega}$.

Instead, one hopes for a localisation of the polymer paths in some $\mathscr{O}(1)$ neighbourhood of some 'favourite' path, expending the least 'energy' H_N^{ω} as depicted in Figure 4. Furthermore, when the inverse temperature $\beta > \beta_c$, we do not expect the polymer endpoint measures to be 'diffusive', i.e.

$$\max_{x\in\mathbb{Z}^d} P_N^{\beta,\omega}(X_N = x) \overset{n\to\infty}{\longrightarrow} 0\,.$$

In dimensions $d = 1, 2$, the existence of this regime has already been established, with $\beta_c = 0$ (that is there is no phase transition). The goal of these notes is to sketch an argument for the existence of the picture as in Figure 3 (and also making the notions of '(de-)localisation' more precise), with a non-trivial critical inverse temperature $\beta_c > 0$. To achieve this, we will be considering for $N \in \mathbb{N}, \beta \in \mathbb{R}$ the renormalised partition functions

$$W_N^{\beta,\omega} = \frac{Z_N^{-\beta,\omega}}{\mathbb{E}Z_N^{-\beta,\omega}}\,,$$

with $Z_N^{-\beta,\omega}$ as in (3.1). we will also be making the standing assumption that for all $\beta \in \mathbb{R}$, namely that the moment generating function of the individual weights in the random environment are always finite, i.e.,

$$\lambda(\beta) \coloneqq \log \mathbb{E}e^{\beta\omega_{\cdot,\cdot}} < \infty\,.$$

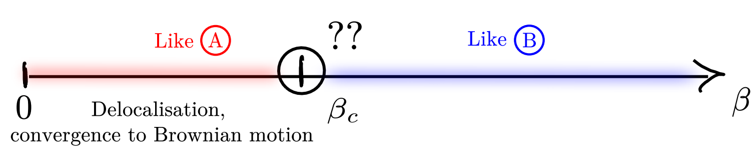

Figure 3: Illustration of expectation for non-trivial phase transition at critical inverse temperature β_c from the 'delocalised regime' (red), to the 'localised' regime (blue).

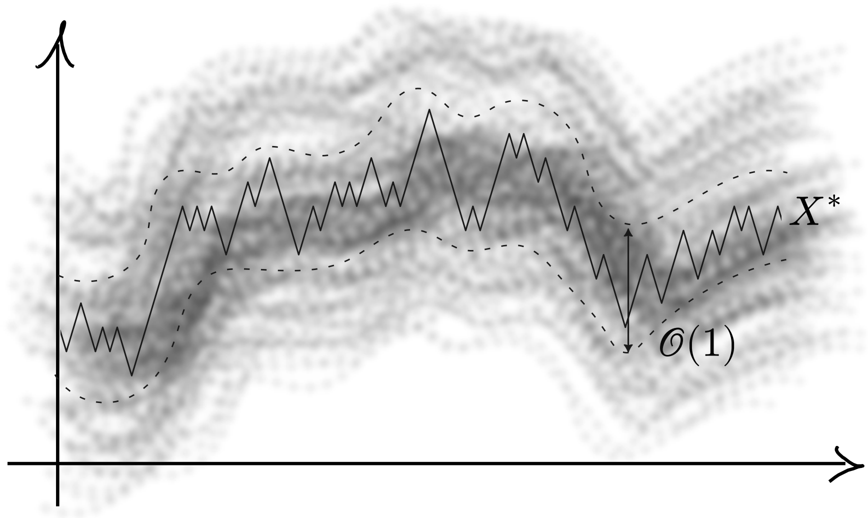

Figure 4: Illustration of concentration of the random polymer measure in an $\mathcal{O}(1)$-neighbourhood of a path X^* with minimal 'energy'.

This makes the expectations of the partition functions well defined. In particular, we can compute

$$\begin{aligned}\mathbb{E}Z_N^{-\beta,\omega} &= \sum_{X\in\Omega_N} \mathbb{E}\mathrm{e}^{\beta H_N^\omega(X)} \\ &= (2d)^N (\mathbb{E}\mathrm{e}^{\beta\omega_{1,0}})^N .\end{aligned}$$

Hence, we can express

$$\begin{aligned}W_N^{\beta,\omega} &= (2d)^{-N} \cdot \sum_{X\in\Omega_N} \exp\left(\beta H_N^\omega(X) - \lambda(\beta)\right) \\ &= E_N\left[\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n} - \lambda(\beta)\right)\right] .\end{aligned}$$

We now state and prove some basic results on the renormalised partition functions $W_N^{\beta,\omega}$, which for ease of notation, drop the explicit dependence on the environment, ω. We begin with a lemma which highlights the underlying martingale structure of the partition function (which are random since they depend on the random environment ω).

Lemma 4.1 ([Bol89])**.** *The renormalised partition functions $(W_N^\beta)_{N\geq 2}$ are a martingale with respect to the filtration $(\mathcal{F}_N)_{N\geq 2}$, where*

$$\mathcal{F}_N \coloneqq \sigma(\{\omega_{n,x} : x \in \mathbb{Z}^d , n \leq N\}) , \qquad N \geq 2 .$$

Proof. For $N \geq 2$, we compute the conditional expectations

$$\begin{aligned}\mathbb{E}[W_{N+1}^\beta|\mathcal{F}_N] &= E_{N+1}\left[\mathbb{E}\left[\exp\left(\sum_{n=1}^{N+1} \beta\omega_{n,X_n} - \lambda(\beta)\right)\middle|\mathcal{F}_N\right]\right] \\ &= E_{N+1}\left[\exp\left(\sum_{n=1}^{N} \beta\omega_{n,X_n} - \lambda(\beta)\right)\cdot \underbrace{\mathbb{E}\left[\exp\left(\beta\omega_{N+1,X_{N+1}} - \lambda(\beta)\right)\middle|\mathcal{F}_N\right]}_{=1}\right] \\ &= W_N^\beta .\end{aligned}$$

□

Thus, being a non-negative martingale, $(W_N^\beta)_{N\geq 2}$ has an almost sure limit as $N\to\infty$, W_∞^β (see the Appendix, 4.2). We now also prove a '$0-1$' law for W_∞^β, which states that the limit can only vanish almost surely or be almost surely positive.

Lemma 4.2. *The limiting renormalised partition function W_∞^β satisfies $\mathbb{P}(W_\infty^\beta = 0)\in\{0,1\}$, and $\mathbb{E}W_\infty^\beta = \mathbb{P}(W_\infty^\beta > 0)$. In other words, $W_\infty^\beta > 0$ almost surely if and only if the family $(W_N^\beta)_{N\geq 2}$ is uniformly integrable.*

Proof. For $z\in\mathbb{Z}^d, k\in\mathbb{N}$, define the shift operators on the environment ω by

$$\Theta_{k,z}\omega \coloneqq (\omega_{n+k,x+z})_{x\in\mathbb{Z}^d,n\in\mathbb{N}}\,,$$

and for any measurable function of the environment define

$$\Theta_{k,z}f(\omega) \coloneqq f\left(\Theta_{k,z}\omega\right)\,.$$

Now, for any $n,N\in\mathbb{N}, x\in\mathbb{Z}^d$, conditioning on the walk at time n, we obtain with

$$\widehat{W}_n^\beta(x) \coloneqq \mathbb{E}\left[\exp\left(\sum_{i=1}^n \beta\omega_{i,X_i} - \lambda(\beta)\right)\cdot \mathbf{1}(X_n = x)\right]\,,$$

$$W_{N+n}^\beta = \sum_{x\in\mathbb{Z}^d} \widehat{W}_N^\beta(x)\cdot \Theta_{N,x}(W_n^\beta)\,.$$

Observe that for any $N\in\mathbb{N}, x\in\mathbb{Z}^d$, $(\Theta_{N,x}(W_n^\beta))_{n\geq 2}$ is a marginale with the same distribution as $(W_n)_{n\geq 2}$ and converges to some limit, $\Theta_{N,x}(W_\infty^\beta)$, almost surely as $n\to\infty$, which has the same distribution as W_∞^β. Thus, taking $n\to\infty$, noting that for fixed $N\in\mathbb{N}$, the sum above (over $x\in\mathbb{Z}^d$) is finite, we obtain using Lemma 4.1,

$$W_\infty^\beta = \sum_{x\in\mathbb{Z}^d} \widehat{W}_N^\beta(x)\cdot \Theta_{N,x}(W_\infty^\beta)\,.$$

Now we obtain the equality of events

$$\{W_\infty^\beta > 0\} = \{\exists x\in Z^d : \Theta_{N,x}(W_\infty^\beta) > 0 \text{ and } \mathbb{P}(X_n = x) > 0\} \in \sigma(\omega_{k,x} : k\geq N+1, x\in Z^d)\,.$$

Since $N\geq 1$ was arbitrary, we deduce that the event $\{W_\infty^\beta > 0\}$ is in the tail sigma algebra of the random environment 'slices', $(\omega_{n,x} : x\in\mathbb{Z}^d)_{n\geq 1}$. Thus, by Kolmogorov's zero-one law, we obtain the first part. For the second part, taking conditional expectations with respect to

$$\mathcal{F}_N \coloneqq \sigma(\omega_{k,x} : 1\leq k\leq N, x\in Z^d))\,,\qquad N\geq 1\,,$$

we see that

$$\mathbb{E}[W_\infty^\beta|\mathcal{F}_N] = \sum_{x\in\mathbb{Z}^d} \widehat{W}_N^\beta(x)\cdot \mathbb{E}[\underbrace{\Theta_{N,x}(W_\infty^\beta)}_{\perp\!\!\!\perp \mathcal{F}_N}|\mathcal{F}_N]\,,$$

which finally gives

$$\mathbb{E}[W_\infty^\beta|\mathcal{F}_N] = W_N^\beta\cdot\mathbb{E}[W_\infty^\beta]\,.$$

Now, by another standard martingale argument applied to the family $(\mathbb{E}[W_\infty^\beta|\mathcal{F}_N])_{N\geq 1}$, which is seen to be a uniformly integrable martingale, we have that it converges almost surely and in L^1 to W_∞^β (see the. Appendix, 4.2), whence we obtain that

$$W_\infty^\beta = W_\infty^\beta\cdot\mathbb{E}[W_\infty^\beta]\,,$$

concluding the proof of the second part. ☐

We now make the following definitions.

Definition 4.3. *With W_∞^β as in Lemma 4.2, if $W_\infty^\beta > 0$ a.s., we say weak disorder holds, and if $W_\infty^\beta = 0$ a.s., we say strong disorder holds.*

In the regime of weak disorder, one can informally think that there is some averaging that occurs, since $W_N^\beta \asymp \mathbb{E}W_N^\beta$, $N \geq 1$ and that moreover, terms in the sums

$$E_N\left[\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n}\right)\right] = \sum_{X\in\Omega_N} \frac{1}{(2d)^N}\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n}\right)$$

contribute equally with

$$\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n}\right) \asymp \mathbb{E}\left[\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n}\right)\right].$$

We now state some propositions from the literature that establish the existence of some phase transition between weak and strong disorder in terms of the inverse temperature parameter β, as well as give existence and some constraints on the critical inverse temperature β_c.

Proposition 4.4 ([CY06]). *There exists some critical $\beta_c = \beta_c(d) \in [0,\infty]$ such that weak disorder holds when $\beta \in \{0\} \cup (0,\beta_c)$ and strong disorder holds when $\beta \in (\beta_c,\infty)$.*

To prove this, it suffices to show the following monotonicity result.

Proposition 4.5. *Fix $N \geq, \beta \in [0,\infty)$ and let $\varphi \in \mathscr{C}^1(\mathbb{R}_+,\mathbb{R})$ with a monotone decreasing derivative. Then, the map*

$$\beta \mapsto \mathbb{E}[W_N^\beta]$$

is monotone decreasing.

Proof. This would follow straightforwardly if one had

$$\frac{\partial}{\partial\beta}\mathbb{E}[\varphi(W_N^\beta)] = \mathbb{E}\left[\frac{\partial}{\partial\beta}\varphi(W_N^\beta)\right] = \mathbb{E}[\varphi'(W_N^\beta)Y_N^\beta], \qquad (*)$$

where

$$Y_N^\beta := E_N\left[\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n} - \lambda(\beta)\right)\cdot\left(\sum_{n=1}^N \omega_{n,X_n} - \lambda'(\beta)\right)\right] \in L^2(\mathbb{P}).$$

Observe that $\varphi'(W_N^\beta)$ is decreasing in the environment ω and that $\sum_{n=1}^N \omega_{n,X_n} - \lambda'(\beta)$ is increasing. Hence, by FKG, Lemma 4.9 and Fubini, we have

$$(*) \leq E_N\left[\mathbb{E}\left[\varphi'(W_N^\beta)\right]\cdot\mathbb{E}\left[\sum_{n=1}^N(\omega_{n,X_n} - \lambda'(\beta))\cdot\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n} - \lambda(\beta)\right)\right]\right] = 0,$$

since the second expectation vanishes. Now, to show that $(*)$ holds, it suffices to observe that for every $\beta > 0$, there exists a neighbourhood $[\beta_1,\beta_2]$, $0 < \beta_1, \beta < \beta_2$ such that

$$\sup_{\beta\in[\beta_1,\beta_2]}\left|\frac{\partial}{\partial\beta}\varphi(W_N^\beta)\right|$$

is integrable. This is essentially to apply the dominated convergence theorem. For the sake of simplicity, we assume that $\varphi'(u) \leq u + u^{-1}$, $u > 0$. We then estimate

$$\left|\frac{\partial}{\partial\beta}\varphi(W_N^\beta)\right| = \varphi'(W_N^\beta)\cdot Y_N^\beta \leq W_n^\beta\cdot Y_N^\beta + (W_N^\beta)^{-1}\cdot Y_N^\beta.$$

Now, we also have

$$\begin{aligned}(W_N^\beta)^{-1} &= \left(E_N\left[\exp\left(\sum_{n=1}^N \beta\omega_{n,X_n} - \lambda(\beta)\right)\right]\right)^{-1} \\ &\le E_N\left[\exp\left(\lambda(\beta) - \sum_{n=1}^N \beta\omega_{n,X_n}\right)\right] \\ &\le \mathrm{e}^{N\lambda(\beta)} \cdot E_N\left[\exp\left(\beta\sum_{n=1}^N |\omega_{n,X_n}|\right)\right].\end{aligned}$$

where the first inequality follows by Jensen applied to $x \mapsto x^{-1}$. We thus obtain

$$\left(\sup_{[\beta_1,\beta_2]} (W_N^\beta)^{-1}\right)^2 \le \left(\sup_{[\beta_1,\beta_2]} \mathrm{e}^{2N\lambda(\beta)}\right) \cdot E_N\left[\exp\left(2\beta_2\sum_{n=1}^N |\omega_{n,X_n}|\right)\right].$$

This gives that

$$\left(\sup_{[\beta_1,\beta_2]} (W_N^\beta)^{-1}\right)^2 \in L^2(\mathbb{P}),$$

which concludes the proof. □

We can now quickly prove Proposition 4.4.

Proof. (Proposition 4.4) Apply Proposition 4.5 with $\varphi(\cdot) \equiv \cdot$ to obtain that the function

$$\beta \mapsto \mathbb{E}[W_\infty^\beta]$$

is non-increasing. This then shows that the set of beta for which weak disorder holds is an interval, whence the result follows. □

In dimensions $d = 1, 2$ it was also established that there is no phase transition. More precisely, the following was proven.

Proposition 4.6 ([CSY03], [CH02]). *In $d = 1, 2$, $\beta_c = 0$.*

In dimensions $d \ge 3$, it was already known that there exists a phase transition.

Proposition 4.7 ([Bol89]). *In $d \ge 3$, $\beta_c > 0$.*

Proof. We will show that for β positive and sufficiently small, the family $(W_N^\beta)_{N\ge 1}$ is bounded in L^2, and so uniformly integrable, which means the phase transition must occur away from zero. Now, notice we can write

$$\begin{aligned}\mathbb{E}[(W_N^\beta)^2] &= \mathbb{E}\left[E_N^{\otimes 2}\left[\exp\left(\textstyle\sum_{n=1}^N \omega_{n,X_n^{(1)}} + \omega_{n,X_n^{(2)}} - 2\lambda(\beta)\right)\right]\right] \\ &\overset{\text{Fubini}}{=} E_N^{\otimes 2}\left[\mathbb{E}\left[\chi(\beta)^{\left(\sum_{n=1}^N \mathbf{1}(X_n^{(1)}=X_n^{(2)})\right)}\right]\right] \\ &\le E_N^{\otimes 2}\left[\mathbb{E}\left[\chi(\beta)^{\left(\sum_{n=1}^\infty \mathbf{1}(X_n^{(1)}=X_n^{(2)})\right)}\right]\right].\end{aligned}$$

where $\chi(\beta) = \mathrm{e}^{\lambda(2\beta)-\lambda(\beta)}$. In $d \ge 3$, $\left(\sum_{n=1}^\infty \mathbf{1}(X_n^{(1)} = X_n^{(2)})\right) < \infty$ is almost surely finite; in fact it is a geometric random variable, by the Strong Markov property. Hence, $\sup_{N\in\mathbb{N}} \mathbb{E}[(W_N^\beta)^2] < \infty$ if $(\chi(\beta) - 1)E^{\otimes^2}\left[\left(\sum_{n=1}^\infty \mathbf{1}(X_n^{(1)} = X_n^{(2)})\right)\right] < 1$, which is achieved taking $\beta > 0$ sufficiently close to zero. □

4.1 Weak disorder implies diffusivity

We now establish the first connection between the weak disorder regime and the 'diffusive' picture in the following theorem, which states that a version of Donsker's theorem holds in this regime. In other words, assuming weak disorder, the 'impurities' in the environment do not affect drastically the macroscopic shape of the polymers.

Theorem 4.8 ([IS88], [Bol89], [AZ96],[SZ96], [CY06])**.** *If weak disorder holds, then one has Theorem 2.1 hold for the rescaled random walks $X^{(N)}$, sampled from the tilted random measures $P_N^{\beta,\omega}$ in probability, that is for any bounded continuous functional $\varphi : C([0,1], \mathbb{R}^d) \to \mathbb{R}$,*

$$E_N \left[\varphi(X^{(N)})\right] \overset{N\to\infty}{\longrightarrow} E_N \left[\varphi(B)\right] ,$$

in probability, where B is a Brownian motion with diffusion matrix $d^{-1}\mathrm{I}_d$.[a]

[a] One can take this convergence to hold ω-almost surely up to a subsequence by a diagonalisation argument.

Before we begin with the proof of the theorem, we first establish some preliminary lemmas. We aim to show that in the weak order regime, the polymer sampled from $P_N^{\beta,\omega}$ look like a concatenation of a polymer sampled from $P_m^{\beta,\omega}$, for $m \leq n$ and an independent simple random walk. That is the polymers macroscopically look like simple random walks (the environent looks uniform macroscopically), and so upon rescaling, one expects a form of Donsker's theorem, Theorem 2.1 to hold. The proof exploits some monotonicity properties of the polymer measures with respect to weights, and for this, we need the following very useful lemma.

Lemma 4.9 (FKG)**.** *Fix $M \geq 1$ and let $X = (X_1, \cdots, X_M)$ be iid real-valued random variables. For $x, y \in \mathbb{R}^M$, write $x \geq y$ if and only if all components satisfy $x_i \geq y_i$, $1 \leq i \leq M$. Now, if $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are such that for $x, y \in \mathbb{R}^M$, $x \geq y$ implies $f(x) \geq f(y)$ and $g(x) \geq g(y)$ (we say f, g are increasing functions), then*

$$\mathbb{E}f(X)g(X) \geq \mathbb{E}f(X) \cdot \mathbb{E}g(X) .$$

Remark. *This is a fundamental inequality which can be generalised to more abstract settings, assuming only one has such an ordering relation on elements of some space which the randomness takes values in. Moreover, if f, g are indicators (i.e. events), this lemma is equivalent to saying that any two increasing events are positively correlated (since they both satisfy the same closure property under majorisation, since for A an increasing event, $x \in A$ implies $y \in A$ for all $y \geq x$).*

Proof. We proceed by induction.
$\underline{M=1}$: Let Y_1 be an independent copy of X_1. Then, by the monotonicity of f, g and considering the cases $X_1 < Y_1$ and $X_1 \geq Y_1$ individually, one can conclude that

$$\begin{aligned} 0 &\leq \mathbb{E}[(F(X_1) - g(X_1)) \cdot (g(X_1) - g(Y_1))] \\ &= 2\mathbb{E}f(X)g(X) - 2\mathbb{E}f(X) \cdot \mathbb{E}g(X) , \end{aligned}$$

from which the desired inequality follows.
$\underline{M \geq 2}$: Now, suppose the inequality is true for $M \geq 2$; we show that it holds for $M+1$. By conditioning on X_1, we see that

$$\mathbb{E}f(X_1, \cdots, X_{M+1})g(X_1, \cdots, X_{M+1}) = \mathrm{Cov}(\underbrace{\mathbb{E}[f(X)|X_1], \mathbb{E}[g(X)|X_1]}_{①}) + \mathbb{E}[\underbrace{\mathrm{Cov}(f(X), g(X)|X_1)}_{②}] .$$

Now, ① ≥ 0 by the base case and conditioning and ② ≥ 0 by the induction hypothesis and conditioning, which concludes the proof. □

We now prove the aforementioned result on the polymer measures looking macroscopically like the simple random walk.

Lemma 4.10. *Let $m \leq n \geq 1$ and define the modified polymer measure on Ω_n by (suppresing explicit dependence on the environment ω)*

$$\tilde{W}_m(\cdot) \coloneqq E_n\left[\exp\left(\sum_{k=1}^{m} \beta\omega_{k,X_k}\right) \cdot \mathbf{1}(X \in \cdot)\right],$$

with normalisation constant $W_m(\mathbf{1}) = \mathbb{E}Z_m^{-\beta,\omega}$. Then, for any measurable function $g : \Omega_n \to [0,1]$, we have

$$\mathbb{E}[|W_n(g) - \tilde{W}_m(g)|] \leq \mathbb{E}[|W_n - W_m|].$$

Proof. By considering the equality $|x| = 2x - x_+$, $x \in \mathbb{R}$, it suffices to show that

$$\mathbb{E}[(W_n(g) - \tilde{W}_m(g))_+] \leq \mathbb{E}[(W_n - W_m)_+].$$

To show this, with $\overline{g} \coloneqq 1 - g$, w have

$$\begin{aligned}(W_n - \tilde{W}_m)_+ &\geq (W_n - \tilde{W}_m)_+ \cdot \mathbf{1}(W_n(g) \geq \tilde{W}_m(g)) \\ &= (W_n - \tilde{W}_m)_+ + (W_n(\overline{g}) - \tilde{W}_m(\overline{g}))_+ \cdot \mathbf{1}(W_n(g) \geq \tilde{W}_m(g)).\end{aligned}$$

Now, observe that conditionally on $\mathcal{F}_m = \sigma(\{\omega_{k,x} : 1 \leq k \leq m, x \in \mathbb{Z}^d\})$, $W_n(\overline{g}) - \tilde{W}_m(\overline{g})$ is an increasing function of $(\omega_{k,x})_{k \geq m, x \in \mathbb{Z}^d}$ (and only depends on a finite number of such indices). Now, by Lemma 4.9 (FKG inequality) and the $\mathcal{F}_m$-measurability of $\tilde{W}_m(\overline{g})$, we obtain the desired conclusion by taking expectations of both sides of the inequality obtained above and using the tower property on the right hand side. ☐

We are now ready to start the proof of Theorem 4.8.

Proof. (Theorem 4.8) For the sake of simplicity, we will assume $d = 1$ and set $\mathscr{C} \coloneqq \mathscr{C}([0,1],\mathbb{R})$. Fix $\varphi \in \mathscr{C}(\mathscr{C},\mathbb{R})$, $N \geq 1$ and let $m = m(N) = N^{\frac{1}{4}}$. For a fixed realisation of the environment ω, we estimate using the triangle equality

$$|E_N^{\beta,\omega}[\varphi(X^{(N)})] - Q(\varphi(B)) \leq \underbrace{|E_N^{\beta,\omega}[\varphi(X^{(N)})] - |\tilde{E}_m^{\beta,\omega}[\varphi(X^{(N)})]|}_{①} + \underbrace{|\tilde{E}_m^{\beta,\omega}[\varphi(X^{(N)})] - Q(\varphi(B))|}_{②},$$

where $\tilde{E}_m^{\beta,\omega}$ is the expectation associated to the polymer measure $\tilde{W}_m$ as in Lemma 4.10 (with $n = N$). We deal with each term separately.
For ①, by Lemma 4.10, we have that

$$\mathbb{E}① \leq \mathbb{E}[|W_n^{\beta,\omega}(\mathbf{1}) - W_m^{\beta,\omega}(\mathbf{1})|] \overset{n,m\to\infty}{\longrightarrow} 0.$$

Now, for the second term, observe that the polymer X sampled according to $\tilde{W}_m^{\beta,\omega}$ is coupled with a random walk $(Y_k)_{k\geq 0}$, such that $|X_k - Y_k| \leq m$, $0 \leq k \leq 2m$ and $(X_k - X_m)_{m \leq k \leq n} = (Y_k - Y_m)_{m \leq k \leq n}$. Hence, by Brownian scaling and the choice of m, one has the estimate

$$\left\|X^{(N)} - Y^{(N)}\right\|_{\infty,[0,1]} \leq \frac{2m}{\sqrt{N}} \leq \frac{2}{N^{\frac{1}{2}}} \overset{N\to\infty}{\longrightarrow} 0.$$

Hence, ② can be estimated (for every realisation of the environment ω), by the triangle inequality as follows,

$$② \leq |\tilde{E}_m^{\beta,\omega}[\varphi(X^{(N)})] - \tilde{E}_m^{\beta,\omega}[\varphi(Y^{(N)})]| + |\mathbb{E}\varphi(Y^{(N)})] - Q(\varphi(B))|.$$

Now, the first term converges to zero as $N \to \infty$ since the rescaled simple random walk paths are tight (by Donsker, Theorem 2.1) and the $X^{(N)}$ being uniformly close are also tight. Now, denote the Banach space of all bounded Lipschitz functionals by

$$\mathrm{BL}(\mathscr{C}, \mathbb{R}) \coloneqq \left\{ F : \mathscr{C} \to \mathbb{R} : \|F\| \coloneqq \|F\|_{\infty,[0,1]} + \sup_{f,g \in \mathscr{C}, f \neq g} \frac{|F(f) - F(g)|}{\|f-g\|_\infty} < \infty \right\}.$$

Now, by [Dud02, Theorem 11.3.3] we have that weak convergence on $\mathscr{C}$ is metrisable with metric given by the Lévy-Prokhorov metric. This means that weak convergence is equivalent to checking convergence in for functionals $\varphi \in \mathrm{BL}(\mathscr{C}, \mathbb{R})$. Clearly, the above coupling on X and Y gives that $X^{(N)}$ converge to Brownian motion on $[0,1]$ and so the first term also vanishes in the limit for all realisations of the environment ω. This in conjunction with the aforementioned L^1 convergence of (1) to zero gives the convergence for a fixed functional, in ω-probability, concluding the proof.
To see how one might prove the footnote in the statement of the theorem, one can use the fact that convergence in probability implies convergence up to a subsequence, and the convergence of (1), (2) is uniform in functionals $\varphi \in \mathrm{BL}(\mathscr{C}, \mathbb{R}), \|\varphi\|_{\mathrm{BL}} \leq M < \infty$. Then, by a diagonal argument, one can extract a subsequence $(N_k)_{k \geq 1}$, such that $\mathbb{E}^{\beta,\omega}_{N_k}[\varphi(X^{(N_k)})], k \geq 1$ converges ω almost surely to $Q(\varphi(B))$ for any $\varphi \in \mathrm{BL}(\mathscr{C}, \mathbb{R})$. Finally, this gives the ω-almost sure convergence using the aforementioned metrisability result of weak convergence involving testing against such bounded Lipschitz functionals. □

Again, to reiterate, we can see that in the weak disorder regime, the inhomogeneities in the random environment 'persist' up to a scale that is not 'seen' by the Brownian scaling, and so the above functional central limit theorem holds.

4.2 very strong disorder

Now, associated to the inverse temperature $\beta \in [0, \infty]$, one can define (formally) the free energy

$$F(\beta) \coloneqq \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log W_N^\beta .$$

We now claim that the limit above exists and is finite almost surely, that is $F(\cdot) \in (-\infty, 0]$. First, observe that the expectations $\mathbb{E}[\frac{1}{N} \log W_N^\beta]$, $N \geq 1$, are super-additive, that is for $n, m \geq 1$,

$$\frac{1}{n+m} \mathbb{E}[\log W_{n+m}^\beta] \geq \frac{1}{n} \mathbb{E}[\log W_n^\beta] + \frac{1}{m} \mathbb{E}[\log W_m^\beta] .$$

Indeed, recall with the notation of Lemma 4.1, we can express the tilted polymer measures

$$W_{n+m}^\beta = \sum_{x \in \mathbb{Z}^d} \widehat{W}_n^\beta(x) \cdot \Theta_{n,x}(W_m^\beta) = W_n \cdot \sum_{x \in \mathbb{Z}^d} \mu_n(x) \cdot \Theta_{n,x}(W_m^\beta) ,$$

with $\mu_N(\cdot) = \mathbb{P}_N^{\beta,\omega}(X_N \in \cdot)$. Thus, taking logarithms gives

$$\log W_{n+m}^\beta = \log W_n + \log \sum_{x \in \mathbb{Z}^d} \mu_n(x) \cdot \Theta_{n,x}(W_m^\beta) \overset{\text{Jensen}}{\geq} \log W_n + \mathbb{E}\left[\sum_{x \in \mathbb{Z}^d} \mu_n(x) \cdot \log \Theta_{n,x}(W_m^\beta) \right]$$

whence the result follows using the fact that $\log \Theta_{n,x}(W_m^\beta) \overset{d}{=} \log W_m^\beta$ for all $n \geq 1, x \in \mathbb{Z}^d$.

This then implies by a standard argument that one has

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\log W_N^\beta] = \sup_{N \geq 1} \frac{1}{N} \mathbb{E}[\log W_N^\beta] \leq 0 .$$

(Super-additive sequences) To quickly see this, let $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a super-additive sequence. suppose first that $\sup_n \frac{b_n}{n} < \infty$. Fix any $\varepsilon > 0$, then there exists some $m \in \mathbb{N}$ such that $\frac{b_m}{m} > \sup_n \frac{b_n}{n} - \varepsilon$. Hence, for any $k \geq m$, we have by Euclidean division that there exists some $q \in \mathbb{Z}_+$ and $r \in [0, m) \cap \mathbb{N}$ such that $k = qm + r$. Thus, the super-additivity of $(b_n)_{n\in\mathbb{N}}$ implies that for $k \geq 1$

$$\begin{aligned}\tfrac{b_k}{k} = \tfrac{b_{qm+r}}{qm+r} &\geq \tfrac{q\cdot b_m + b_r}{qm+r} \\ &\geq \underbrace{\tfrac{qm}{qm+r}}_{\to 1} \sup_n \frac{b_n}{n} - \varepsilon \cdot mq + \underbrace{\frac{b_r}{qm+r}}_{\to 0}\end{aligned}$$

as $k \to \infty$. The case where $\sup_n \frac{b_n}{n} = \infty$ can be dealt with similarly (replacing $\sup_N b_n/n - \varepsilon$ for $\varepsilon > 0$ with $N \geq 1$).

Note further that the free energy $F(\cdot)$ is continuous (using Hölder to obtain the convexity of $\beta \mapsto F(\beta) + \lambda(\beta)$) and non-increasing. To see the latter, one can apply Proposition 4.5 with $\varphi \equiv \log$ to obtain that for every $N \geq 1$,

$$\beta \mapsto \frac{1}{N}\mathbb{E}[\log W_N^\beta]$$

is decreasing. Then taking $N \to \infty$, gives the desired monotonicity for the free energy, $F(\cdot)$.

All of the above give the picture in Figure 5 for for the qualitative profile of the free energy. Then it follows that there exists some inverse temperature (possibly infinite) $\overline{\beta}_c = \inf\{\beta \geq 0 : F(\beta) > 0\}$, such that on $\{0\} \cup (0, \overline{\beta}_c)$, $F(\cdot)$ vanishes.

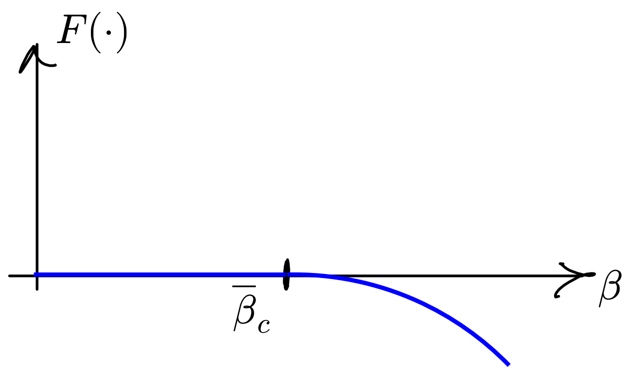

Figure 5: Illustration of free energy $F(\cdot)$ associated to the 'macroscopic' profile of directed polymers in a random environment.

One can show that under certain conditions on the moment log-moment generating function $\lambda(\cdot)$, $F(\cdot)$ is strictly negative, which implies a phase transition in this sense happens. Indeed, we show this by further estimating the free energy as

$$F(\beta) \leq \liminf_{N\to\infty} \frac{2}{N} \log \mathbb{E}[\sqrt{W_N}] < \infty\,.$$

Indeed, observe for the first inequality that for $N \geq 1$, $\mathbb{E}[\log W_N] = 2\mathbb{E}[\log \sqrt{W_N}] \leq 2\log \mathbb{E}[\sqrt{W_N}]$, where the last inequality follows by Jensen. One can further estimate the above bound by observing that using the elementary inequality

$$\sqrt{\sum_{i=1}^n a_i} \leq \sum_{i=1}^n \sqrt{a_i}\,, \qquad a_i \geq 0\,, 1 \leq i \leq n\,, n \geq 1\,,$$

one can estimate (pointwise ω-a.s. and taking expectations)

$$\begin{aligned}\mathbb{E}[\sqrt{W_N}] &\leq \tfrac{1}{(2d)^{\frac{N}{2}}}\mathbb{E}\left[\sum_{X\in\Omega_N}\exp\left(\sum_{n=1}^{N}\frac{\beta}{2}\omega_{n,X_n}-\frac{\lambda(\beta)}{2}\right)\right]\\ &\leq (2d)^{\frac{N}{2}}\left(\exp\left(\lambda(\beta/2)-\tfrac{\lambda(\beta)}{2}\right)\right)^{N}.\end{aligned}$$

Now, if the exponent $\lambda(\beta/2)-\lambda(\beta)/2\to-\infty$, as $\beta\to\infty$, then one is in the strong disorder regime. This leads to the following notion of disorder, which we now discuss.

Definition 4.11. *For $\beta>\overline{\beta}_c$, say that very strong disorder holds.*[a]

[a]Note this name is indeed justified since by the above remark W_N^{β} converges to zero as $N\to\infty$ almost surely.

Recall the definition of the polymer endpoint measures, for $N\geq 1,\beta\in[0,\infty]$, $\mu_N^{\beta}\coloneqq P_N^{\beta}(X_N\in\cdot)$, and set

$$I_n\coloneqq\sum_{x\in\mathbb{Z}^d}(\mu_n(x))^2\,.$$

One can relate the above functionals I_n, $n\geq 1$ to the free energy $F(\cdot)$ as follows. First, observe that by a direct computation using the definition of W_N^{β} in a certain range of inverse temperatures β in the following way.

Theorem 4.12 ([CH02, CSY03])**.** *For any $0<\beta_1<\beta<\beta_2<\infty$, there exist $m,M\in(0,\infty)$ such that for such that for all $\beta\in[\beta_1,\beta_2]$,*

$$m|F(\beta)|\leq\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}I_n\leq\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}I_n\leq M|F(\beta)|\,.$$

Proof. Recall that for $n\geq 1$ and with $\mu_n(\cdot)=P_{n-1}^{\beta,\omega}(X_n\in\cdot)$ the tilted polymer endpoint measure, one can express $P_{n-1}^{\beta,\omega}(X_n\in\cdot)=\mathscr{D}\mu_{n-1}$, where $\mathscr{D}$ is the transition kernel of the simple random walk. Observe that we have the estimates

$$\frac{1}{2d}I_{n-1}\leq\sum_{x\in\mathbb{Z}^d}(\mathscr{D}\mu_{n-1}(x))^2\leq I_{n-1}\,.$$

Now, let $(\mathcal{F}_n)_{n\geq 0}=\sigma(\{\omega_{k,x}:x\in\mathbb{Z}^d,1\leq k\leq n\})$, then we can express for $N\geq 1$,

$$\begin{aligned}\log W_N &=\sum_{n=1}^{N}\log\frac{W_n}{W_{n-1}}\\ &=\underbrace{\sum_{n=1}^{N}\log\frac{W_n}{W_{n-1}}-\mathbb{E}\left[\log\frac{W_n}{W_{n-1}}\middle|\mathcal{F}_{n-1}\right]}_{A_N}+\underbrace{\sum_{n=1}^{N}\mathbb{E}\left[\log\frac{W_n}{W_{n-1}}\middle|\mathcal{F}_{n-1}\right]}_{B_N}.\end{aligned}$$

This is essentially a decompotision of the increments of the free energy into a martingale part A_N and a predictable part B_N.

Recall again that we can represent for $n\geq 1$,

$$W_n=\sum_{x\in\mathbb{Z}^d}\widehat{W}_{n-1}(x)\cdot\Theta_{n-1,x}W_1\,.$$

Hence, we can express the ratio

$$\frac{W_n}{W_{n-1}} = \sum_{x\in\mathbb{Z}^d} \mu_{n-1}(x)\cdot \underbrace{\Theta_{n-1,x}W_1}_{=(\mathscr{D}\zeta_\beta)(n,x)}\,,$$

where we set $\zeta_\beta(n,x) \coloneqq \mathrm{e}^{\beta\omega_{n,X_n}-\lambda(\beta)}$. This can also be expressed as

$$\frac{W_n}{W_{n-1}} = \sum_{x\in\mathbb{Z}^d} \mathscr{D}\mu_{n-1}(x)\cdot \zeta_\beta(n,x)\,,$$

and to taking logarithms and setting $\overline{\zeta}_\beta(n,x) = \zeta_\beta(n,x)-1$, we have

$$\log\frac{W_n}{W_{n-1}} = \log\left(1+\sum_{x\in\mathbb{Z}^d} \mathscr{D}\mu_{n-1}(x)\cdot \overline{\zeta}_\beta(n,x)\right)\,.$$

Moreover, observe that we have the uniform bounds in $n,\geq 1\,, x\in\mathbb{Z}^d$, for $\zeta_\beta(n,x)$ defined above, namely $\frac{1}{K_\beta}\leq \zeta_\beta(n,x)\leq K_\beta$, for some $K_\beta>0$. This implies that

$$\sum_{x\in\mathbb{Z}^d} \mathscr{D}\mu_{n-1}(x)\cdot \overline{\zeta}_\beta(n,x) \in \left[-1+\frac{1}{K_\beta}, K_\beta\right]\,.$$

This gives the uniform boundedness of the martingale increments $(A_N - A_{N-1})_{N\geq 1}$, and so by Azuma-Hoeffding inequality, we have the folowing concentration estimate,

$$\mathbb{P}(A_N \geq N^{\frac{3}{4}}) \leq \exp(-CN^{\frac{1}{2}})\,,$$

for some positive constant $C>0$. Thus, by Borel-Cantelli lemma, we have that $A_N/N\to 0$ almost surely as $N\to\infty$.

It now remains to estimate B_N, $N\geq 1$. To do this, observe that, in $[-1+\frac{1}{K_\beta}, K_\beta]$, one has the following estimates on the logarithm

$$-M_\beta\cdot y^2 \leq \log(1+y) \leq -m_\beta\cdot y^2\,, \qquad y\in\left[-1+\frac{1}{K_\beta}, K_\beta\right]\,,$$

for some positive constants $m_\beta, M_\beta>0$. Using this in the expression for B_N, we obtain for $n\geq 1$,

$$\begin{aligned}
&\mathbb{E}\left[\log\left(1+\sum_{x\in\mathbb{Z}^d} \mathscr{D}\mu_{n-1}(x)\cdot \overline{\zeta}_\beta(n,x)\right) - \sum_{x\in\mathbb{Z}^d} \mathscr{D}\mu_{n-1}(x)\cdot \overline{\zeta}_\beta(n,x)\middle|\mathcal{F}_{n-1}\right]\\
&\asymp -\mathbb{E}\left[\left(\sum_{x\in\mathbb{Z}^d} \mathscr{D}\mu_{n-1}(x)\cdot \overline{\zeta}_\beta(n,x)\right)^2\middle|\mathcal{F}_{n-1}\right]\\
&= -\mathrm{Var}(\zeta_\beta(1,0))\cdot \textstyle\sum_{x\in\mathbb{Z}^d}(\mathscr{D}\mu_{n-1}(x))^2\\
&\asymp -I_{n-1}\,,
\end{aligned}$$

where the variance is bounded from below and using the bounds on the generator $\mathscr{D}$ of the SRW. The upper bound is entirely analogous (using the aforementioned bounds on the logarithm), this gives the desired result upon taking Césaro means of the inequalities thus obtained, concluding the proof. ☐

This in particular gives the following result in the very strong disorder regime.

Theorem 4.13 ([CH02], [CSY03])**.** *For $\beta \in [0,\infty)$, if $F(\beta) < 0$, then*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} I_n > 0\,.^{a}$$

[a] Note when $\beta = 0$, μ_N^β is the uniform measure, and so one obtains the limit below vanishes by direct computation.

Now, all of the above implies the landscape for polymer fluctuations in Figure 6. The fact that the phase transitions for very strong disorder and weak disorders are not separated (that is the gap in Figure 6 collapses to a point), is a consequence of the following theorem.

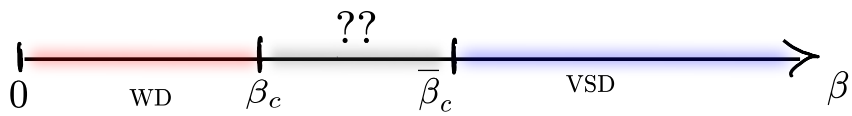

Figure 6: Illustration of the landscape of regimes of polymer fluctuations with respect to the inverse temperature β. The fact that the critical parameters $\beta_c \le \overline{\beta}_c$ (here depicted as a strict inequality) for weak (WD) and very strong disorders (VSD) follow from the remarks made before.

Theorem 4.14 ([JL24], Theorem 2.1)**.** *Assuming the environment has almost surely bounded weights, very strong disorder is equivalent to strong disorder. In particular*

(1) $\beta_c = \overline{\beta}_c$,

(2) $W_\infty^{\beta_c} > 0$ *almost surely.*

In particular, point (2) implies that at criticality, one is in the regime of weak disorder.

To prove (1), it suffices to show that $\beta_c \ge \overline{\beta}_c$, since the converse inequality follows directly from definitions. The fact taht this is inded the case is the content of the following theorem.

Theorem 4.15 ([JL24])**.** *Assuming boundedness of the weights ω, then if $W_N \to 0$, as $N \to \infty$ almost surely, then $F(\beta) < 0$.*

Proof. The result will follow if we can show that if for some $N \ge 1$, $\mathbb{E}[\sqrt{W_N^\beta}] < (2N+1)^{-1}$, then $\mathbb{E}[\sqrt{W_{mN}^\beta}]$ converges exponentially fast in $m \ge 1$ to zero. Indeed, this would imply using the aforementioned bound on the free energy

$$F(\beta) \le \liminf_{N\to\infty} \frac{2}{N} \log \mathbb{E}\left[\sqrt{W_N^\beta}\right] ,$$

that the subsequence $\frac{2}{mN}\log\mathbb{E}\left[\sqrt{W_{mN}^\beta}\right]$ eventually descends below some negative threshold, which would give the result.

To show the aforementioned decay of the partition functions at the more 'macroscopic' scale $(mN)_{m\ge 1}$, we will use a 'coarse-graining' of the polymer sample paths of length mN into m time steps (see Figure 7) and the 'multiplicative' structure of the polymer partition functions. Now, we can express the partition function W^β_{mN}, $m \geq 1$ as

$$W^\beta_{mN} = \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \mathbb{E}\left[\exp\left(\sum_{k=1}^{mN} \beta\omega_{k,X_k} - \lambda(\beta)\right) \cdot \prod_{j=1}^{m} \mathbf{1}(X_{jN} = x_j)\right]$$
$$= \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \prod_{j=1}^{m} \Theta_{(i-1)N,x_{i-1}} \widehat{W}^\beta_N(x_j - x_{j-1}),$$

and taking square roots gives

$$\sqrt{W_{mn}} \leq \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \prod_{j=1}^{m} \sqrt{\Theta_{(i-1)N,x_{i-1}} \widehat{W}^\beta_N(x_j - x_{j-1})}\,.$$

Now, by independence, taking expectations gives

$$\mathbb{E}[\sqrt{W_{mn}}] \leq \sum_{x_1,\cdots,x_m \in \mathbb{Z}^d} \prod_{j=1}^{m} \mathbb{E}\left[\sqrt{\widehat{W}^\beta_N(x_j - x_{j-1})}\right].$$

Now, observe that by $|x_i - x_{i-1}| \leq N$ for $1 \leq i \leq m$, we have (using $\widehat{W}^\beta_N(x_j - x_{j-1}) \leq W^\beta_N$, $j \geq 1$)

$$\mathbb{E}[\sqrt{W_{mn}}] \leq \left(\mathbb{E}\left[(2N+1)^d) \cdot \sqrt{W^\beta_N}\right]\right)^m.$$

Thus, it suffices to show that $\mathbb{E}[\sqrt{W^\beta_N}] < (2N+1)^{-d}$. To show, this, first consider the 'size-biased' measure on the environment, $\tilde{P}^\beta_N$, defined by

$$\mathrm{d}\tilde{P}^\beta_N = W^\beta_N \,\mathrm{d}P\,.$$

Informally, this measure corresponds to 'tilting' the environment by the partition function W^β_N, favouring larger weights in the environment locally. Now, for any A measurable,

$$\mathbb{E}\left[\sqrt{W^\beta_N}\right] = \mathbb{E}\left[\sqrt{W^\beta_N} \cdot \mathbf{1}_A\right] + \mathbb{E}\left[\sqrt{W^\beta_N} \cdot \mathbf{1}_{A^c}\right],$$

whence, we obtain upon applying by Cauchy-Schwarz and Jensen,

$$\mathbb{E}\left[\sqrt{W^\beta_N} \cdot \mathbf{1}_A\right] \leq \sqrt{\mathbb{P}^\beta_N(A)} + \sqrt{\tilde{\mathbb{P}}^\beta_N(A^c)}\,.$$

Hence, we would be done if we can show that there exists some event A such that both $\mathbb{P}^\beta_N(A)$ and $\tilde{\mathbb{P}}^\beta_N(A^c)$ are small. In order to see that this is indeed the case, we need some preliminary lemmas.
We first start with a tight (up to a constant proportionality factor) maximal inequality for the supremum of the polymer partition functions, in the strong disorder regime.

Lemma 4.16. *If $W^\beta_\infty = 0$ then,*

$$\mathbb{P}\left(\sup_{N\geq 1} W_N \geq u\right) \in \left[\frac{1}{Lu}, \frac{1}{u}\right], \qquad u > 0\,,$$

for some constant $L > 0$.

Proof. Since the environment ω is almost surely bounded, we have that there exists some $L > 0$ such that for any $n \geq 1$, $W_n/W_{n-1} \leq L$ (using the bound from Theorem 4.12 proof). Now, for $u > 0$, define the stopping time

$$\tau_u \coloneqq \inf\{n \geq 1 : W_n \geq u\} .$$

Then, observe that by the optional stopping theorem for bounded the martingale $(W_{n\wedge\tau_u})_{n\geq 0}$, we have

$$\mathbb{E}[W_{\tau_u \wedge n}] = 1 ,$$

for all $n \geq 1$. Now, taking $n \to \infty$, the dominated convergence theorem gives

$$\mathbb{E}[W_{\tau_u} \cdot \mathbf{1}_{\{\tau_u < \infty\}}] = 1 .$$

Hence, we have the estimate

$$u \cdot \mathbb{P}(\tau_u < \infty) \leq Lu \cdot \mathbb{P}(\tau_u < \infty) ,$$

where the upper bound follows from thefact that on the event $\{\tau_u < \infty\}$, $W_{\tau_u} \leq LW_{\tau_u - 1} \leq Lu$ almost surely. □

We now state the second lemma, which is an anti-concentration result on the tails of polymer partition functions, which we state without proof.

Lemma 4.17 ([JL24], Proposition 4.2)**.** *Given $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that for all $u > 1$, there exists some $m_\varepsilon \in [\![1, C_\varepsilon \log u]\!]$ such that $\mathbb{P}(W_{m_u}) \geq u^{-1-\varepsilon}$.*

Informally, the above lemma says that (by inspecting the tight lower bound on the tail of the maximal polymer partition function value) the maximal value of the polymer partition functions is attained 'quickly' (in a logarithmic scale).
We thus obtain the following estimates on polymer partition function tail events for $m \geq 1, u \geq 1$ using Markov's inequality and the above lemma.

$$\begin{aligned} \mathbb{P}(W_m \geq u) &\leq \tfrac{1}{u} \\ \tilde{\mathbb{P}}_m(W_m \geq u) &= \mathbb{E}[W_m \cdot \mathbf{1}_{\{W_m \geq u\}}] \geq \tfrac{1}{u^\varepsilon} . \end{aligned}$$

Now, armed with the above results, for $N \geq 1$, with $u = N^{4d}$, $m = m_\varepsilon$ as in the above lemma for $\varepsilon = 1/(12d)$, we define the event

$$\mathscr{A}_N \coloneqq \{\exists (k,x) \in [\![0,N]\!] \times [\![-N,N]\!]^d, \Theta_{k,x} W_m \geq N^{4d}\} .$$

In other words, there exists some location in space-time where the polymer partition function (started from that location) attains a large value. Now, we estimate by a union bound

$$\mathbb{P}_N^\beta(\mathscr{A}_N) \leq (N+1) \cdot (2N+1)^d \cdot \mathbb{P}(W_m \geq N^{4d}) \leq CN^{1-3d} ,$$

for some constant $C > 0$. The above estimates give that

$$\tilde{\mathbb{P}}_m(W_m \geq N^{4d}) \geq N^{-4d\varepsilon} ,$$

and using the inclusion

$$\mathscr{A}_N^c = \bigcap_{(k,x)\in[\![0,N]\!]\times[\![-N,N]\!]^d} \{\Theta_{k,x} W_m \leq N^{4d}\} \subseteq \bigcap_{(k,x)\in[\![0,N/m-1]\!]\times[\![-N,N]\!]^d} \{\Theta_{k,x} W_m \leq N^{4d}\} ,$$

we estimate the size-biased probability of the complement event using the 'multiplicative' structure of the polymer partition functions and hence of the tilted measures $\tilde{\mathbb{P}}$, as

$$\tilde{\mathbb{P}}_N^\beta(\mathscr{A}_N^c) \leq \left(\tilde{\mathbb{P}}_m(W_m \leq N^{4d})\right)^{N/m} \leq \left(1 - N^{-1/3}\right)^{N/m} \leq \exp(-C' N^{1/2}) ,$$

for some constant $C' > 0$. Thus, we have shown that both $\mathbb{P}_N^\beta(\mathscr{A}_N)$ and $\tilde{\mathbb{P}}_N^\beta(\mathscr{A}_N^c)$ decay to zero as $N \to \infty$, concluding the proof. □

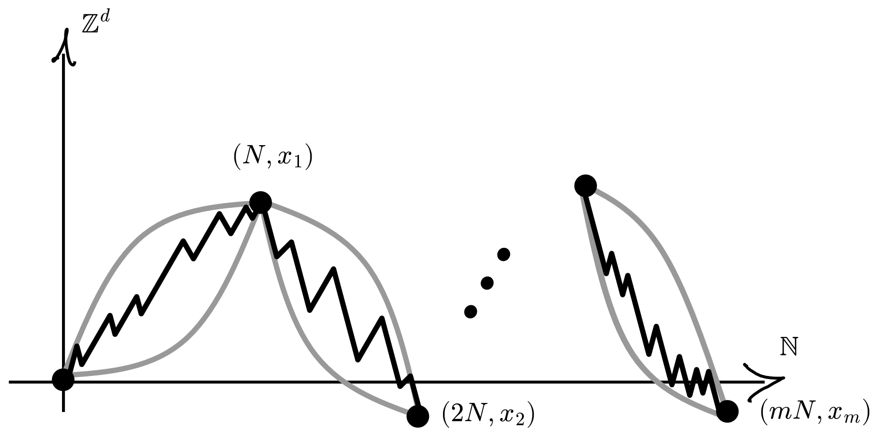

Figure 7: Illutration of coarse-graining of polymer paths in the proof of Theorem 4.15 into blocks of size N to obtain the decay of partition functions at scale mN, $m \geq 1$.

Moreover, one can obtain strong regualrity of the free energy near the critical inverse temperature β_c in the following sense.

Theorem 4.18 ([JL24]). *One has*

$$\lim_{\beta \searrow \beta_c} \frac{\log |F(\beta - \beta_c)|}{\log |\beta - \beta_c|} = \infty \, .$$

In particular, this implies for all $k \geq 1$, $|F(\beta)| \leq C_k \cdot (\beta - \beta_c)^k$, for $\beta \in [\beta_c, \beta_c + 1]$.

Appendix – Martingale convergence

For completeness, we give a brief overview of the almost sure martingale convergence theorem in discrete time, which we used in the proof of Theorem 4.15. Here, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$, i.e. an increasing sequence of sub-σ-algebras of $\mathcal{F}$. We recall the definition of a (super)martingale.

Definition 4.19. *A real-valued stochastic process $X = (X_n)_{n\geq 0}$ adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is called a supermartingale if for all $n \geq 0$:*

(1) $X_n \in L^1(\mathbb{P})$,

(2) X_n *is $\mathcal{F}_n$-measurable,*

(3) $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$, *a.s.*

If the inequality in (3) is an equality, then X is called a martingale.

Theorem 4.20 (Almost sure martingale convergence theorem). *Let X be a supermartingale bounded in L^1, i.e. satisfying $\sup_n \mathbb{E}[|X_n|] < \infty$. Then, there exists $X_\infty \in L^1(\mathcal{F}_\infty), \mathcal{F}_\infty = \sigma(\mathcal{F}_n : n \geq 0)$ such that $X_n \overset{n\to\infty}{\longrightarrow} X_\infty$, a.s.*

Before we embark on the proof of this theorem, we need so me supporting results. First we have a result from analysis and we set up some notation. Let $x = (x_n)_{n\in\mathbb{N}}$ be a real sequence and let $a < b$ be reals. We proceed to define the *number of upcrossings of the sequence* before time $n \in \mathbb{N}$. We c construct recursively the sequence of times:

$$\begin{aligned} T_0(x) &= 0 \\ S_{k+1}(x) &= \inf\{n \geq T_k(x) : x_n \leq a\} \\ T_{k+1}(x) &= \inf\{n \geq S_{k+1}(x) : x_n \geq b\} \end{aligned}$$

and

$$N_n([a,b],X) = \sup\{k \geq 0 : T_k(x) \leq n\}$$

Observe that as $n \to \infty$, $N_n([a,b],x) \uparrow N([a,b],x) = \sup\{k \geq 0 : T_k(x) < \infty\}$ (see figure 8 for an illustration).

Lemma 4.21. *Let $X = (X_n)$ be a real sequence. Then X converges in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ if and on ly if for all $a < b$, $a, b \in \mathbb{Q}$, $N([a,b],X) < \infty$.*

Proof. $\underline{\Longrightarrow :}$ Suppose x converges, if $a < b$ such that $N([a,b],x) = \infty$, then $\liminf_n x_n \leq a < b \leq \limsup_n x_n$, a contradiction.
$\underline{\Longleftarrow :}$ if not, then $\liminf_n x_n < \limsup_n x_n$ which implies that there exists $a < b$ in $\mathbb{Q}$ with $\liminf_n x_n < a < b < \limsup_n x_n$, and hence $N([a,n],x) = \infty$, a contradiction. □

Now we state it Doob's upcrossing Inequality

Lemma 4.22 (Doob's upcrossing inequality)**.** *Let X be a supermartingale, then for all $n \in \mathbb{N}$:*

$$(b-a)\cdot\mathbb{E}[N_n([a,b],X)] \leq \mathbb{E}[(X_n - a)^-]$$

Proof. It is not hard to check that the sequences of times in 4.2 are stopping times. Now we have:

$$\begin{aligned} &\sum_{k=1}^{n}(X_{T_k\wedge n} - X_{S_k\wedge n}) \\ &= \underbrace{\sum_{k=1}^{N_n}(X_{T_k} - X_{S_k})}_{\geq N_n\cdot(b-a)} + (X_n - X_{S_{N_n+1}})\mathbf{1}(S_{N_n+1} \leq n) \end{aligned}$$

Since $T_{k\wedge n} \geq S_{k\wedge n}$, the optional stopping theorem (OST) gives $\mathbb{E}[X_{T_k\wedge n}] \leq \mathbb{E}[X_{S_k\wedge n}]$. Note:

$$\underbrace{X_n - X_{S_{N_n}+1}}_{\geq (X_n-a)\wedge 0 = -(X_n-a)^-} \quad \mathbf{1}(S_{N_n+1} \leq n).$$

Thus, taking expectations on both sides gives:

$$0 \geq (b-a)\cdot\mathbb{E}[N_n] - \mathbb{E}[(X_n-a)^-].$$

thus concluding the proof. □

Now we proceed to the proof of the martingale convergence theorem:

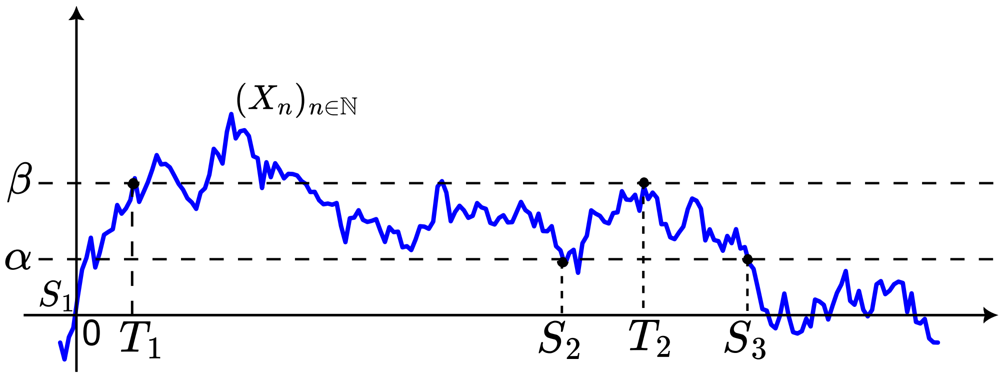

Figure 8: Illustration of upcrossings for the process $(X_n)_{n\in\mathbb{N}}$.

Proof. (Theorem 4.20) Fix $a < b$, in $\mathbb{Q}$. Have

$$\begin{aligned}\mathbb{E}[N_{n([a,b],X)}] &\leq (b-a)^{-} \underbrace{\mathbb{E}[(X_n - a)^{-}]}_{\leq \mathbb{E}[|X_n|+a]} \\ &\leq (b-a)^{-}\left(\sup_{n\geq 0} \underbrace{\mathbb{E}[|X_n|]}_{<\infty} + a\right)\end{aligned}$$

Also have $N_n([a,b],X) \uparrow N([a,b],X)$ as $n \to \infty$. By monotone convergence: $\mathbb{E}[N([a,b],X)] < \infty$. Set

$$\Omega_0 = \bigcap_{a<b,a,b,\in\mathbb{Q}} \{N([a,b],X) < \infty\} \in \mathcal{F}_\infty$$

and $\mathbb{P}(\Omega_0) = 1$. On Ω_0, X converges. set

$$X_\infty = \begin{cases} \lim\limits_{n\to\infty} X_n & \text{on } \Omega_0 \\ 0, & \text{on } \Omega \setminus \Omega_0. \end{cases}$$

So, X_∞ is $\mathcal{F}_\infty$−measurable, $X_n \overset{n\to\infty}{\longrightarrow} X_\infty$ a.s. and

$$\mathbb{E}[|X_\infty|] = \mathbb{E}[\liminf_n |X_n|] \leq \liminf_{\mathbb{E}[X_n]} < \infty.$$

□

Corollary 4.23. *Let B be a supermartingale. Then, X converges a.s.*

Proof. $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \leq \mathbb{E}[X_0]$. Apply the martingale convergence theorem to conclude. □

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